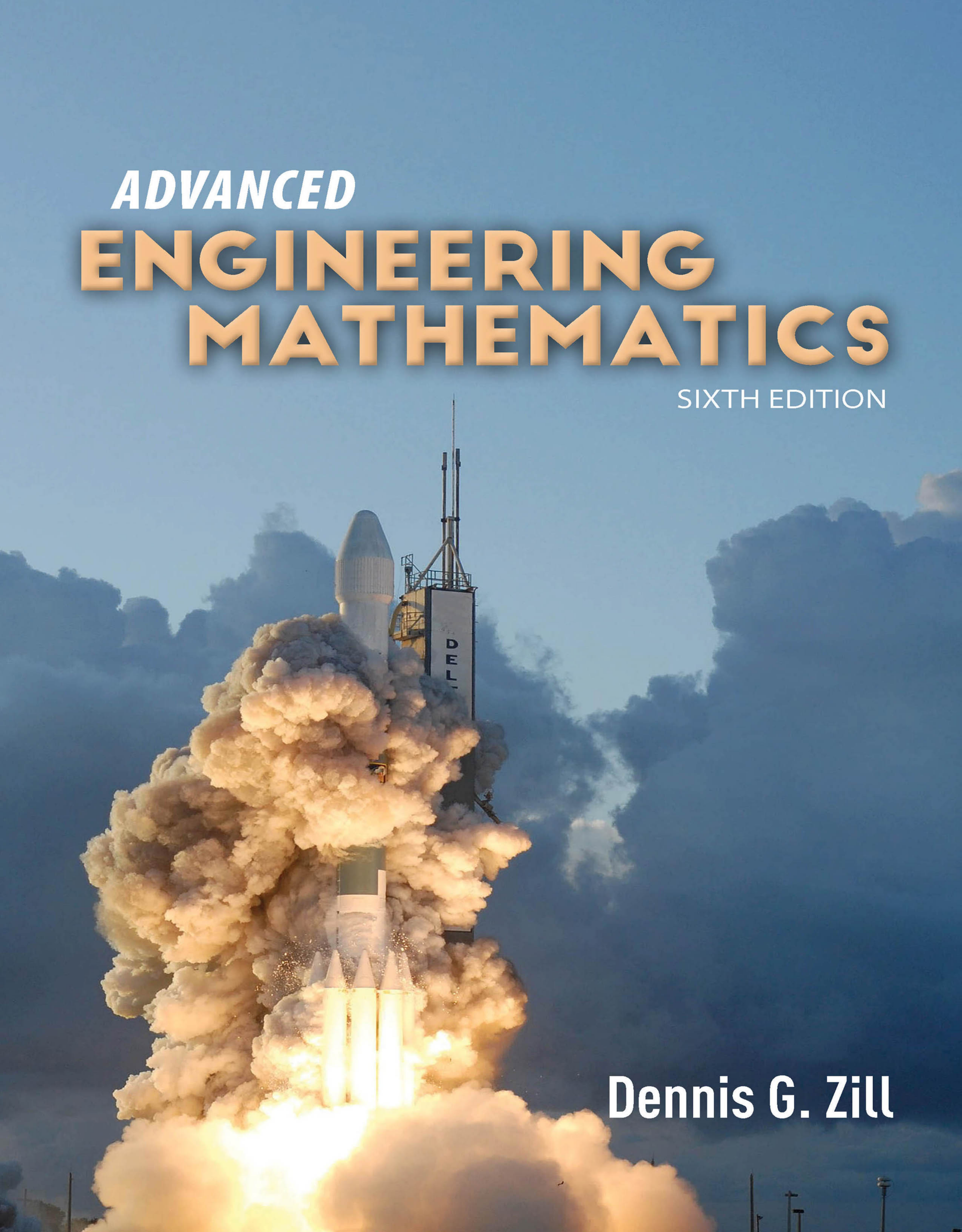
# ADVANCED ENGINEERING MATHEMATICS

SIXTH EDITION



Dennis G. Zill

# Welcome

## Differentiation Rules

1. Constant: $\frac{d}{dx} c = 0$

2. Constant Multiple: $\frac{d}{dx} cf(x) = c f'(x)$

3. Sum: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

4. Product: $\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$

5. Quotient: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

6. Chain: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

7. Power: $\frac{d}{dx} x^n = nx^{n-1}$

8. Power: $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$

## Derivatives of Functions

Trigonometric:

9. $\frac{d}{dx}\sin x = \cos x$

10. $\frac{d}{dx}\cos x = -\sin x$

11. $\frac{d}{dx}\tan x = \sec^2 x$

12. $\frac{d}{dx}\cot x = -\csc^2 x$

13. $\frac{d}{dx}\sec x = \sec x \tan x$

14. $\frac{d}{dx}\csc x = -\csc x \cot x$

Inverse trigonometric:

15. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$

16. $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$

17. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$

18. $\frac{d}{dx}\cot^{-1} x = -\frac{1}{1 + x^2}$

19. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}$

20. $\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2 - 1}}$

Hyperbolic:

21. $\frac{d}{dx}\sinh x = \cosh x$

22. $\frac{d}{dx}\cosh x = \sinh x$

23. $\frac{d}{dx}\tanh x = \text{sech}^2 x$

24. $\frac{d}{dx}\coth x = -\text{csch}^2 x$

25. $\frac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x$

26. $\frac{d}{dx}\text{csch}\, x = -\text{csch}\, x \coth x$

Inverse hyperbolic:

27. $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$

28. $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$

29. $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1 - x^2},\ |x| < 1$

30. $\frac{d}{dx}\coth^{-1} x = \frac{1}{1 - x^2},\ |x| > 1$

31. $\frac{d}{dx}\text{sech}^{-1} x = -\frac{1}{x\sqrt{1 - x^2}}$

32. $\frac{d}{dx}\text{csch}^{-1} x = -\frac{1}{|x|\sqrt{x^2 + 1}}$

Exponential:

33. $\frac{d}{dx} e^x = e^x$

34. $\frac{d}{dx} b^x = b^x(\ln b)$

Logarithmic:

35. $\frac{d}{dx}\ln|x| = \frac{1}{x}$

36. $\frac{d}{dx}\log_b x = \frac{1}{x(\ln b)}$

Of an integral:

37. $\frac{d}{dx}\int_a^x g(t)\, dt = g(x)$

38. $\frac{d}{dx}\int_a^b g(x, t)\, dt = \int_a^b \frac{\partial}{\partial x} g(x, t)\, dt$

## Integration Formulas

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

3. $\displaystyle\int e^u\,du = e^u + C$

4. $\displaystyle\int b^u\,du = \frac{1}{\ln b}b^u + C$

5. $\displaystyle\int \sin u\,du = -\cos u + C$

6. $\displaystyle\int \cos u\,du = \sin u + C$

7. $\displaystyle\int \sec^2 u\,du = \tan u + C$

8. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

9. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

10. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

11. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$

12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$

15. $\displaystyle\int u \sin u\,du = \sin u - u\cos u + C$

16. $\displaystyle\int u \cos u\,du = \cos u + u\sin u + C$

17. $\displaystyle\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

18. $\displaystyle\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

19. $\displaystyle\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

20. $\displaystyle\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

21. $\displaystyle\int e^{au} \sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

22. $\displaystyle\int e^{au} \cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

23. $\displaystyle\int \sinh u\,du = \cosh u + C$

24. $\displaystyle\int \cosh u\,du = \sinh u + C$

25. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

26. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

27. $\displaystyle\int \tanh u\,du = \ln(\cosh u) + C$

28. $\displaystyle\int \coth u\,du = \ln|\sinh u| + C$

29. $\displaystyle\int \ln u\,du = u\ln u - u + C$

30. $\displaystyle\int u \ln u\,du = \tfrac{1}{2}u^2 \ln u - \tfrac{1}{4}u^2 + C$

31. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C$

32. $\displaystyle\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left|u + \sqrt{a^2+u^2}\right| + C$

33. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

34. $\displaystyle\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2+u^2}\right| + C$

35. $\displaystyle\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\ln\left|\frac{a+u}{a-u}\right| + C$

36. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

37. $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left|u + \sqrt{u^2-a^2}\right| + C$

38. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2-a^2}\right| + C$

# ADVANCED ENGINEERING MATHEMATICS

SIXTH EDITION

**Dennis G. Zill**
Loyola Marymount University

JONES & BARTLETT
LEARNING

*World Headquarters*
Jones & Bartlett Learning
5 Wall Street
Burlington, MA 01803
978-443-5000
info@jblearning.com
www.jblearning.com

Jones & Bartlett Learning books and products are available through most bookstores and online booksellers. To contact Jones & Bartlett Learning directly, call 800-832-0034, fax 978-443-8000, or visit our website, www.jblearning.com.

Substantial discounts on bulk quantities of Jones & Bartlett Learning publications are available to corporations, professional associations, and other qualified organizations. For details and specific discount information, contact the special sales department at Jones & Bartlett Learning via the above contact information or send an email to specialsales@jblearning.com.



**Production Credits**
VP, Executive Publisher: David D. Cella
Executive Editor: Matt Kane
Acquisitions Editor: Laura Pagluica
Associate Editor: Taylor Ferracane
Vendor Manager: Sara Kelly
Director of Marketing: Andrea DeFronzo
VP, Manufacturing and Inventory Control: Therese Connell
Composition and Project Management: Aptara®, Inc.
Cover Design: Kristin E. Parker
Rights & Media Specialist: Merideth Tumasz
Media Development Editor: Shannon Sheehan
Cover Images: Domestic: © NASA International: © CHEN MIN CHUN/Shutterstock
Printing and Binding: RR Donnelley
Cover Printing: RR Donnelley

**To order this product, use ISBN: 978-1-284-10590-2**

**Library of Congress Cataloging-in-Publication Data**
Author: Zill, Dennis G.
Title: Advanced Engineering Mathematics / Dennis G. Zill, Loyola Marymount University.
Description: Sixth edition. | Burlington, MA : Jones & Bartlett Learning, [2017] | Includes index.
Identifiers: LCCN 2016022410| ISBN 9781284105902 (casebound) | ISBN 1284105903 (casebound)
Subjects: LCSH: Engineering mathematics.
Classification: LCC TA330 .Z55 2017 | DDC 620.001/51—dc23
LC record available at https://lccn.loc.gov/2016022410

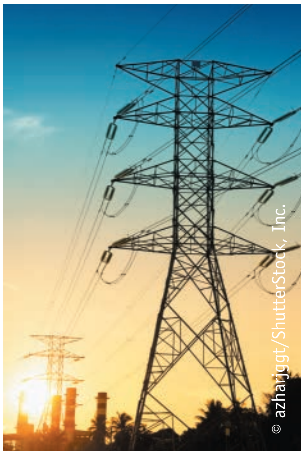
6048

Printed in the United States of America
20 19 18 17 16 10 9 8 7 6 5 4 3 2 1


# Contents


A380
© Tim Jenner/ShutterStock, Inc.




© azhar/ShutterStock, Inc.


© Cecilia Lim H M/ShutterStock, Inc.




© sixninepixels/Shutterstock, Inc.

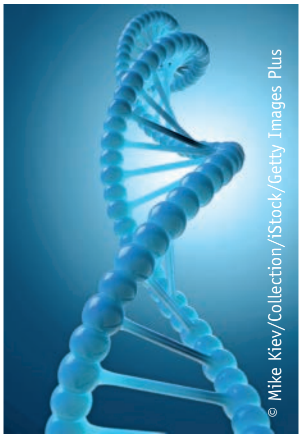



© Vaclav Volrab/Shutterstock, Inc.


© Image State/Alamy Stock Photo




© Dennis Hallinan/Alamy Images


© Mike Kiev/Collection/iStock/Getty Images Plus




© kochanowski/Shutterstock


© enigmatico/ShutterStock, Inc.




© Design Pics Inc./Alamy Images



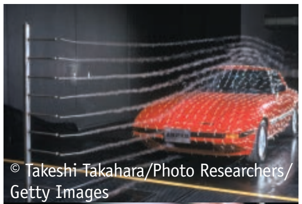

© Peteri/ShutterStock, Inc.


© hofhauser/Shutterstock, Inc.

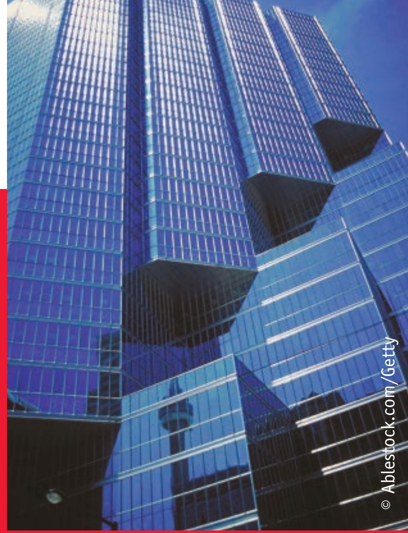



Dennis K. Johnson/Getty Images




© Takeshi Takahara/Photo Researchers/
Getty Images


© Andy Sacks/Getty Images




# Preface

In courses such as *calculus* or *differential equations*, the content is fairly standardized but the content of a course entitled *engineering mathematics* often varies considerably between two different academic institutions. Therefore a text entitled *Advanced Engineering Mathematics* is a compendium of many mathematical topics, all of which are loosely related by the expedient of either being needed or useful in courses in science and engineering or in subsequent careers in these areas. There is literally no upper bound to the number of topics that could be included in a text such as this. Consequently, this book represents the author's opinion of what constitutes *engineering mathematics*.

## Content of the Text

For flexibility in topic selection this text is divided into five major parts. As can be seen from the titles of these various parts it should be obvious that it is my belief that the backbone of science/engineering related mathematics is the theory and applications of ordinary and partial differential equations.

### Part 1: Ordinary Differential Equations (Chapters 1–6)

The six chapters in Part 1 constitute a complete short course in ordinary differential equations. These chapters, with some modifications, correspond to Chapters 1, 2, 3, 4, 5, 6, 7, and 9 in the text *A First Course in Differential Equations with Modeling Applications*, *Eleventh Edition*, by Dennis G. Zill (Cengage Learning). In Chapter 2 the focus is on methods for solving first-order differential equations and their applications. Chapter 3 deals mainly with linear second-order differential equations and their applications. Chapter 4 is devoted to the solution of differential equations and systems of differential equations by the important Laplace transform.

### Part 2: Vectors, Matrices, and Vector Calculus (Chapters 7–9)

Chapter 7, *Vectors*, and Chapter 9, *Vector Calculus*, include the standard topics that are usually covered in the third semester of a calculus sequence: vectors in 2- and 3-space, vector functions, directional derivatives, line integrals, double and triple integrals, surface integrals, Green's theorem, Stokes' theorem, and the divergence theorem. In Section 7.6 the vector concept is generalized; by defining vectors analytically we lose their geometric interpretation but keep many of their properties in $n$-dimensional and infinite-dimensional vector spaces. Chapter 8, *Matrices*, is an introduction to systems of algebraic equations, determinants, and matrix algebra, with special emphasis on those types of matrices that are useful in solving systems of linear differential equations. Optional sections on cryptography, error correcting codes, the method of least squares, and discrete compartmental models are presented as applications of matrix algebra.

### Part 3: Systems of Differential Equations (Chapters 10 and 11)

There are two chapters in Part 3. Chapter 10, *Systems of Linear Differential Equations*, and Chapter 11, *Systems of Nonlinear Differential Equations*, draw heavily on the matrix material presented in Chapter 8 of Part 2. In Chapter 10, systems of linear first-order equations are solved utilizing the concepts of eigenvalues and eigenvectors, diagonalization, and by means of a matrix exponential function. In Chapter 11, qualitative aspects of autonomous linear and nonlinear systems are considered in depth.

### Part 4: Partial Differential Equations (Chapters 12–16)

The core material on Fourier series and boundary-value problems involving second-order partial differential equations was originally drawn from the text *Differential Equations with Boundary-Value Problems*, *Ninth Edition*, by Dennis G. Zill (Cengage Learning). In Chapter 12, *Orthogonal Functions and Fourier Series*, the fundamental topics of sets of orthogonal functions and expansions of functions in terms of an infinite series of orthogonal functions are presented. These topics are then utilized in Chapters 13 and 14 where boundary-value problems in rectangular, polar, cylindrical, and spherical coordinates are solved using the method of separation of variables. In Chapter 15, *Integral Transform Method*, boundary-value problems are solved by means of the Laplace and Fourier integral transforms.

### Part 5: Complex Analysis (Chapters 17–20)

The final four chapters of the hardbound text cover topics ranging from the basic complex number system through applications of conformal mappings in the solution of Dirichlet's problem. This material by itself could easily serve as a one quarter introductory course in complex variables. This material was taken from *Complex Analysis*: *A First Course with Applications*, *Third Edition*, by Dennis G. Zill and Patrick D. Shanahan (Jones & Bartlett Learning).

### Additional Online Material: Probability and Statistics (Chapters 21 and 22)

These final two chapters cover the basic rudiments of probability and statistics and can obtained as either a PDF download on the accompanying Student Companion Website and Projects Center or as part of a custom publication. For more information on how to access these additional chapters, please contact your Account Specialist at go.jblearning.com/findmyrep.

## Design of the Text

For the benefit of those instructors and students who have not used the preceding edition, a word about the design of the text is in order. Each chapter opens with its own table of contents and a brief introduction to the material covered in that chapter. Because of the great number of figures, definitions, and theorems throughout this text, I use a double-decimal numeration system. For example, the interpretation of "Figure 1.2.3" is

Chapter Section of Chapter 1
↓ ↓
1.2.3 ← Third figure in Section 1.2

I think that this kind of numeration makes it easier to find, say, a theorem or figure when it is referred to in a later section or chapter. In addition, to better link a figure with the text, the *first* textual reference to each figure is done in the same font style and color as the figure number. For example, the first reference to the second figure in Section 5.7 is given as **FIGURE 5.7.2** and all subsequent references to that figure are written in the tradition style Figure 5.7.2.

## Key Features of the Sixth Edition

- The principal goal of this revision was to add many new, and I feel interesting, problems and applications throughout the text. For example, *Sawing Wood* in Exercises 2.8, *Bending of a Circular Plate* in Exercises 3.6, *Spring Pendulum* in Chapter 3 in Review, and *Cooling Fin* in Exercises 5.3 are new to this edition. Also, the application problems

  *Air Exchange*, Exercises 2.7
  *Potassium*-40 *Decay*, Exercises 2.9
  *Potassium-Argon Dating*, Exercises 2.9
  *Invasion of the Marine Toads*, Chapter 2 in Review
  *Temperature of a Fluid*, Exercises 3.6
  *Blowing in the Wind*, Exercises 3.9
  *The Caught Pendulum*, Exercises 3.11
  *The Paris Guns*, Chapter 3 in Review

  contributed to the last edition were left in place.
- Throughout the text I have given a greater emphasis to the concepts of piecewise-linear differential equations and solutions that involve integral-defined functions.
- The superposition principle has been added to the discussion in Section 13.4, *Wave Equation*.
- To improve its clarity, Section 13.6, *Nonhomogeneous Boundary-Value Problems*, has been rewritten.
- Modified Bessel functions are given a greater emphasis in Section 14.2, *Cylindrical Coordinates*.

## Supplements

***For Instructors***

- *Complete Solutions Manual* (*CSM*) by Warren S. Wright and Roberto Martinez
- Test Bank
- Slides in PowerPoint format
- Image Bank
- WebAssign: WebAssign is a flexible and fully customizable online instructional system that puts powerful tools in the hands of teachers, enabling them to deploy assignments, instantly assess individual student performance, and realize their teaching goals. Much more than just a homework grading system, WebAssign delivers secure online testing, customizable precoded questions directly from exercises in this textbook, and unparalleled customer service. Instructors who adopt this program for their classroom use will have access to a digital version of this textbook. Students who purchase an access code for WebAssign will also have access to the digital version of the printed text.

  With WebAssign instructors can:
  - Create and distribute algorithmic assignments using questions specific to this textbook
  - Grade, record, and analyze student responses and performance instantly
  - Offer more practice exercises, quizzes, and homework
  - Upload resources to share and communicate with students seamlessly

For more detailed information and to sign up for free faculty access, please visit webassign.com. For information on how students can purchase access to WebAssign bundled with this textbook, please contact your Jones and Bartlett account representative at go.jblearning.com/findmyrep.

Designated instructor materials are for qualified instructors only. Jones & Bartlett Learning reserves the right to evaluate all requests. For detailed information and to request access to instructor resources, please visit go.jblearning.com/ZillAEM6e.

***For Students***

- A WebAssign Student Access Code can be bundled with a copy of this text at a discount when requested by the adopting instructor. It may also be purchased separately online when WebAssign is required by the student's instructor or institution. The student access code provides the student with access to his or her specific classroom assignments in WebAssign and access to a digital version of this text.
- A *Student Solutions Manual* (*SSM*) prepared by Warren S. Wright and Roberto Martinez provides a solution to every third problem from the text.
- Access to the Student Companion Website and Projects Center, available at go.jblearning.com/ZillAEM6e, is included with each new copy of the text. This site includes the following resources to enhance student learning:
    - Chapter 21 Probability
    - Chapter 22 Statistics
    - Additional projects and essays that appeared in earlier editions of this text, including:

*Two Properties of the Sphere*
*Vibration Control*: *Vibration Isolation*
*Vibration Control*: *Vibration Absorbers*
*Minimal Surfaces*
*Road Mirages*
*Two Ports in Electrical Circuits*
*The Hydrogen Atom*
*Instabilities of Numerical Methods*
*A Matrix Model for Environmental Life Cycle Assessment*
*Steady Transonic Flow Past Thin Airfoils*
*Making Waves*: *Convection*, *Diffusion*, *and Traffic Flow*
*When Differential Equations Invaded Geometry*: *Inverse Tangent Problem of the 17th Century*
*Tricky Time*: *The Isochrones of Huygens and Leibniz*
*The Uncertainty Inequality in Signal Processing*
*Traffic Flow*
*Temperature Dependence of Resistivity*
*Fraunhofer Diffraction by a Circular Aperture*
*The Collapse of the Tacoma Narrow Bridge*: *A Modern Viewpoint*
*Atmospheric Drag and the Decay of Satellite Orbits*
*Forebody Drag of Bluff Bodies*

## Acknowledgments

The task of compiling a text this size is, to say the least, difficult and many people have put much time and energy into this revision. So I would like to take this opportunity to express my sincerest appreciation to everyone—most of them unknown to me—at Jones & Bartlett Learning and at Aptara, Inc. who were involved in the publication of this

edition. A special word of thanks goes to my editor Laura Pagluica and production editor Sherrill Redd for their guidance in putting all the pieces of a large puzzle together.

Over the years I have been very fortunate to receive valuable input, solicited and unsolicited, from students and my academic colleagues. An occasional word of support is always appreciated, but it is the criticisms and suggestions for improvement that have enhanced each edition. So it is fitting that I once again recognize and thank the following reviewers for sharing their expertise and insights:

Raul M. Aguilar
*Massachusetts Maritime Academy*

A. Alton
*Augustana University*

Yuri Antipov
*Louisiana State University*

Victor Argueta
*Alma College*

Ken Bosworth
*Idaho State University*

Kristen Campbell
*Elgin Community College*

Han-Taw Chen
*National Cheng Kung University*

John T. Van Cleve
*Jacksonville State University*

William Criminale
*University of Washington*

Juan F. Diaz, Jr.
*Mount Aloysius College*

Vlad Dobrushkin
*University of Rhode Island*

Jeff Dodd
*Jacksonville State University*

Victor Elias
*University of Western Ontario*

Robert E. Fennell
*Clemson University*

Seferino Fierroz
*Oxbridge Academy*

Stan Freidlander
*Bronx Community University*

David Gilliam
*Texas Tech University*

Stewart Goldenberg
*California Polytechnic State University*

Herman Gollwitzer
*Drexel University*

Ronald B. Gunther
*Oregon State University*

Daniel Hallinan, Jr.
*Florida A&M University–Florida State University College of Engineering*

Noel Harbetson
*California State University*

Angela Hare
*Messiah College*

Donald Hartig
*California Polytechnic State University*

Sonia Henckel
*Lawrence Technological University*

Robert W. Hunt
*Humbolt State University*

David Keyes
*Columbia University*

Mario Klaric
*Midlands Technical College*

Vuryl Klassen
*California State University, Fullerton*

Cecilia Knoll
*Florida Institute of Technology*

Myren Krom
*California State University, Sacramento*

David O. Lomen
*University of Arizona*

Maria Ludu
*Embry-Riddle Aeronautical University*

Lewis D. Ludwig
*Denison University*

Tony Mastroberardino
*Penn State Erie, The Behrend College*

Osvaldo Mendez
*University of Texas, El Paso*

Kelley B. Mohrmann
*U.S. Military Academy*

James L. Moseley
*West Virginia University*

Gregory E. Muleski
*University of Missouri, Kansas City*

Charles P. Neumann
*Carnegie Mellon University*

Evgeni Nikolaev
*Rutgers University*

Bruce O'Neill
*Milwaukee School of Engineering*

Sang June Oh
*California State University, Fullerton*

Dale Peterson
*US Air Force Academy*

Christopher S. Raymond
*University of Delaware*

Geoffrey Recktenwald
*Michigan State University*

Thomas N. Roe
*South Dakota State University*

Gary Stout
*Indiana University of Pennsylvania*

Jeremy L. Thompson
*US Air Force Academy*

Benjamin Varela
*Rochester Institute of Technology*

Tian-Shiang Yang
*National Cheng Kung University*

Bashkim Zendeli
*Lawrence Technological University*

I also wish to express my sincerest gratitude to the following individuals who were kind enough to contribute applied problems to this edition:

Jeff Dodd, Professor, Department of Mathematical Sciences,
*Jacksonville State University*, Jacksonville, Alabama

Pierre Gharghouri, Professor Emeritus, Department of Mathematics,
*Ryerson University*, Toronto, Canada

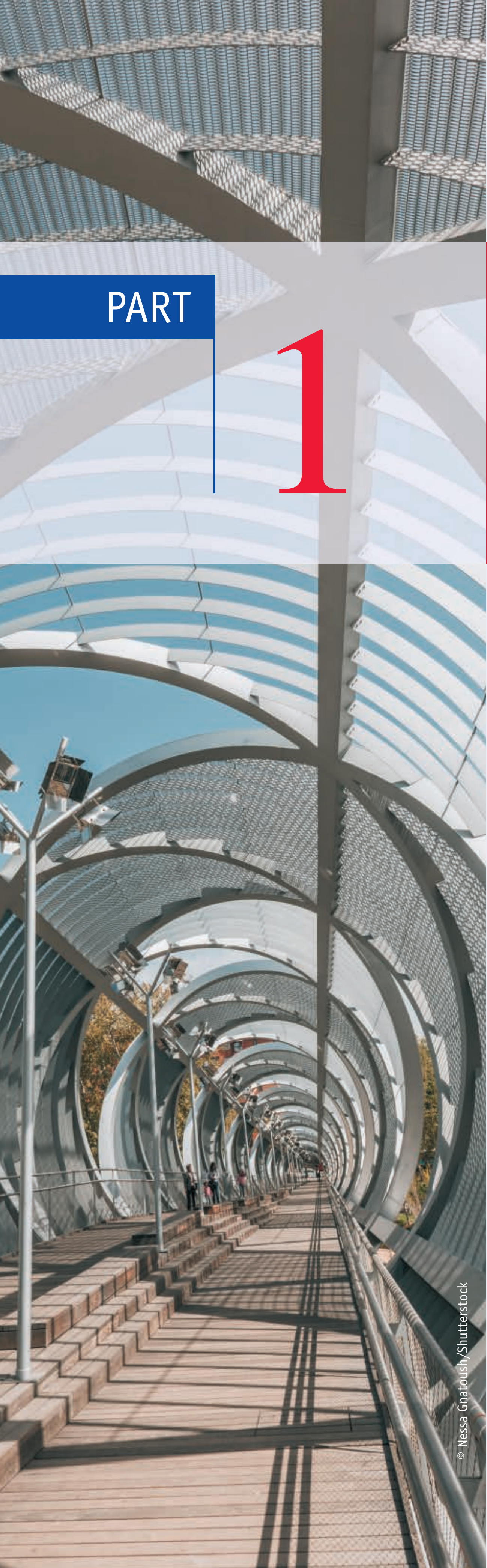
Jean-Paul Pascal, Associate Professor, Department of Mathematics,
*Ryerson University*, Toronto, Canada

Rick Wicklin, PhD, Senior Researcher in Computational Statistics,
*SAS Institute Inc.*, Cary, North Carolina

Although many eyes have scanned the thousands of symbols and hundreds of equations in the text, it is a surety that some errors persist. I apologize for this in advance and I would certainly appreciate hearing about any errors that you may find, either in the text proper or in the supplemental manuals. In order to expedite their correction, contact my editor at:

LPagluica@jblearning.com

Dennis G. Zill





PART 1

# Ordinary Differential Equations

CHAPTER

# 1

# Introduction to Differential Equations

The purpose of this short chapter is twofold: to introduce the basic terminology of **differential equations** and to briefly examine how differential equations arise in an attempt to describe or **model** physical phenomena in mathematical terms.

## CHAPTER CONTENTS

# 1.1 Definitions and Terminology

**INTRODUCTION** The words *differential* and *equation* certainly suggest solving some kind of equation that contains derivatives. But before you start solving anything, you must learn some of the basic defintions and terminology of the subject.

## A Definition

The derivative $dy/dx$ of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. For example, the function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$, and its derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ in the last equation by the symbol $y$, we obtain

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

Now imagine that a friend of yours simply hands you the **differential equation** in (1), and that you have no idea how it was constructed. Your friend asks: "What is the function represented by the symbol $y$?" You are now face-to-face with *one* of the basic problems in a course in differential equations:

*How do you solve such an equation for the unknown function $y = \phi(x)$?*

The problem is loosely equivalent to the familiar reverse problem of differential calculus: Given a derivative, find an antiderivative.

Before proceeding any further, let us give a more precise definition of the concept of a differential equation.

**Definition 1.1.1 Differential Equation**

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

In order to talk about them, we will classify a differential equation by **type**, **order**, and **linearity**.

## Classification by Type

If a differential equation contains only ordinary derivatives of one or more functions with respect to a *single* independent variable it is said to be an **ordinary differential equation (ODE)**. An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **partial differential equation (PDE)**. Our first example illustrates several of each type of differential equation.

**EXAMPLE 1** Types of Differential Equations

**(a)** The equations

an ODE can contain more than one dependent variable ↓ ↓

$$\frac{dy}{dx} + 6y = e^{-x}, \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y \tag{2}$$

are examples of ordinary differential equations.

**(b)** The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

are examples of partial differential equations. Notice in the third equation that there are two dependent variables and two independent variables in the PDE. This indicates that $u$ and $v$ must be functions of *two or more* independent variables. ≡

**Notation** Throughout this text, ordinary derivatives will be written using either the **Leibniz notation** $dy/dx$, $d^2y/dx^2$, $d^3y/dx^3$, . . . , or the **prime notation** $y'$, $y''$, $y'''$, . . . . Using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 6y = e^{-x}$ and $y'' + y' - 12y = 0$, respectively. Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of $y''''$. In general, the $n$th derivative is $d^ny/dx^n$ or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the differential equation $d^2x/dt^2 + 16x = 0$, it is immediately seen that the symbol $x$ now represents a dependent variable, whereas the independent variable is $t$. You should also be aware that in physical sciences and engineering, **Newton's dot notation** (derogatively referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time $t$. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, the first and second equations in (3) can be written, in turn, as $u_{xx} + u_{yy} = 0$ and $u_{xx} = u_{tt} - u_t$.

**Classification by Order** The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation.

### EXAMPLE 2 Order of a Differential Equation

The differential equations

$$\underset{\text{highest order}}{\frac{d^2y}{dx^2}} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x, \qquad 2\underset{\text{highest order}}{\frac{\partial^4 u}{\partial x^4}} + \frac{\partial^2 u}{\partial t^2} = 0$$

are examples of a second-order ordinary differential equation and a fourth-order partial differential equation, respectively. ≡

A first-order ordinary differential equation is sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

### EXAMPLE 3 Differential Form of a First-Order ODE

If we assume that $y$ is the dependent variable in a first-order ODE, then recall from calculus that the differential $dy$ is defined to be $dy = y'\,dx$.

**(a)** By dividing by the differential $dx$ an alternative form of the equation $(y - x)\,dx + 4x\,dy = 0$ is given by

$$y - x + 4x\frac{dy}{dx} = 0 \quad \text{or equivalently} \quad 4x\frac{dy}{dx} + y = x.$$

**(b)** By multiplying the differential equation

$$6xy\frac{dy}{dx} + x^2 + y^2 = 0$$

by $dx$ we see that the equation has the alternative differential form

$$(x^2 + y^2)\,dx + 6xy\,dy = 0.$$ ≡

In symbols, we can express an $n$th-order ordinary differential equation in one dependent variable by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{4}$$

where $F$ is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons, we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where $f$ is a real-valued continuous function, is referred to as the **normal form** of (4). Thus, when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2 y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations.

## EXAMPLE 4 Normal Form of an ODE

**(a)** By solving for the derivative $dy/dx$ the normal form of the first-order differential equation

$$4x\frac{dy}{dx} + y = x \quad \text{is} \quad \frac{dy}{dx} = \frac{x - y}{4x}.$$

**(b)** By solving for the derivative $y''$ the normal form of the second-order differential equation

$$y'' - y' + 6y = 0 \quad \text{is} \quad y'' = y' - 6y.$$

≡

## Classification by Linearity

An $n$th-order ordinary differential equation (4) is said to be **linear** in the variable $y$ if $F$ is linear in $y, y', \ldots, y^{(n)}$. This means that an $n$th-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{6}$$

Two important special cases of (6) are **linear first-order** ($n = 1$) and **linear second-order** ($n = 2$) ODEs.

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{7}$$

In the additive combination on the left-hand side of (6) we see that the characteristic two properties of a linear ODE are

Remember these two characteristics of a linear ODE.

- The dependent variable $y$ and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree; that is, the power of each term involving $y$ is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable $x$.

A **nonlinear** ordinary differential equation is simply one that is not linear. If the coefficients of $y, y', \ldots, y^{(n)}$ contain the dependent variable $y$ or its derivatives or if powers of $y, y', \ldots, y^{(n)}$, such as $(y')^2$, appear in the equation, then the DE is nonlinear. Also, nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$ cannot appear in a linear equation.

## EXAMPLE 5 Linear and Nonlinear Differential Equations

**(a)** The equations

$$(y - x)dx + 4x\,dy = 0, \quad y'' - 2y' + y = 0, \quad x^3\frac{d^3 y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$$

are, in turn, examples of *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated in part (a) of Example 3 that the first equation is linear in $y$ by writing it in the alternative form $4xy' + y = x$.

**(b)** The equations

$$\underset{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y\\ \downarrow}}{(1-y)}y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underset{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y\\ \downarrow}}{\sin y} = 0, \qquad \frac{d^4y}{dx^4} + \underset{\substack{\text{nonlinear term:}\\ \text{power not 1}\\ \downarrow}}{y^2} = 0,$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively. ≡

## Solution

As stated before, one of our goals in this course is to solve—or find solutions of—differential equations. The concept of a solution of an ordinary differential equation is defined next.

**Definition 1.1.2 Solution of an ODE**

Any function $\phi$, defined on an interval $I$ and possessing at least $n$ derivatives that are continuous on $I$, which when substituted into an $n$th-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an $n$th-order ordinary differential equation (4) is a function $\phi$ that possesses at least $n$ derivatives and

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \text{ for all } x \text{ in } I.$$

We say that $\phi$ *satisfies* the differential equation on $I$. For our purposes, we shall also assume that a solution $\phi$ is a real-valued function. In our initial discussion we have already seen that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally it will be convenient to denote a solution by the alternative symbol $y(x)$.

## Interval of Definition

You can't think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval $I$ in Definition 1.1.2 is variously called the **interval of definition**, the **interval of validity**, or the **domain of the solution** and can be an open interval $(a, b)$, a closed interval $[a, b]$, an infinite interval $(a, \infty)$, and so on.

### EXAMPLE 6 Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

**(a)** $\dfrac{dy}{dx} = xy^{1/2}; \quad y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0; \quad y = xe^x$

**SOLUTION** One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every $x$ in the interval $(-\infty, \infty)$.

**(a)** From *left-hand side*: $\dfrac{dy}{dx} = 4 \cdot \dfrac{x^3}{16} = \dfrac{x^3}{4}$

*right-hand side*: $xy^{1/2} = x \cdot \left(\dfrac{x^4}{16}\right)^{1/2} = x \cdot \dfrac{x^2}{4} = \dfrac{x^3}{4}$,

we see that each side of the equation is the same for every real number $x$. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

**(b)** From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have for every real number $x$,

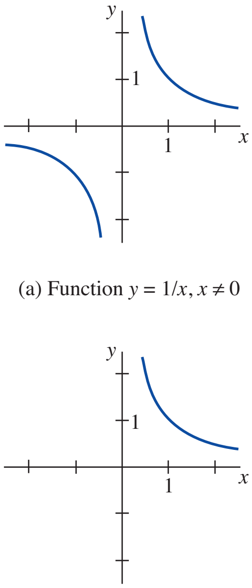
*left-hand side*: $y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$

*right-hand side*: $0$. ≡

Note, too, that in Example 6 each differential equation possesses the constant solution $y = 0$, defined on $(-\infty, \infty)$. A solution of a differential equation that is identically zero on an interval $I$ is said to be a **trivial solution**.

## Solution Curve

The graph of a solution $\phi$ of an ODE is called a **solution curve**. Since $\phi$ is a differentiable function, it is continuous on its interval $I$ of definition. Thus there may be a difference between the graph of the *function* $\phi$ and the graph of the *solution* $\phi$. Put another way, the domain of the function $\phi$ does not need to be the same as the interval $I$ of definition (or domain) of the solution $\phi$.




(a) Function $y = 1/x, x \neq 0$

(b) Solution $y = 1/x, (0, \infty)$

**FIGURE 1.1.1** Example 7 illustrates the difference between the function $y = 1/x$ and the solution $y = 1/x$

### EXAMPLE 7 Function vs. Solution

**(a)** Considered simply as a *function*, the domain of $y = 1/x$ is the set of all real numbers $x$ except 0. When we graph $y = 1/x$, we plot points in the $xy$-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in **FIGURE 1.1.1(a)**. The function $y = 1/x$ is not differentiable at $x = 0$ since the $y$-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

**(b)** Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$ (verify). But when we say $y = 1/x$ is a *solution* of this DE we mean it is a function defined on an interval $I$ on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval not containing 0, such as $(-3, -1)$, $(\frac{1}{2}, 10)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ on the intervals $(-3, -1)$ and on $(\frac{1}{2}, 10)$ are simply segments or pieces of the solution curves defined by $y = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$, respectively, it makes sense to take the interval $I$ to be as large as possible. Thus we would take $I$ to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on the interval $(0, \infty)$ is shown in Figure 1.1.1(b). ≡

## Explicit and Implicit Solutions

You should be familiar with the terms *explicit* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**. For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. We shall see when we get down to the business of actually solving some ordinary differential equations that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when attempting to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution $\phi$ implicitly.

**Definition 1.1.3 Implicit Solution of an ODE**

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval $I$ provided there exists at least one function $\phi$ that satisfies the relation as well as the differential equation on $I$.

It is beyond the scope of this course to investigate the conditions under which a relation $G(x, y) = 0$ defines a differentiable function $\phi$. So we shall assume that if the formal implementation of a method of solution leads to a relation $G(x, y) = 0$, then there exists at least one function $\phi$ that satisfies both the relation (that is, $G(x, \phi(x)) = 0$) and the differential equation on an interval $I$. If the implicit solution $G(x, y) = 0$ is fairly simple, we may be able to solve for $y$ in terms of $x$ and obtain one or more explicit solutions. See (*iv*) in the *Remarks*.

### EXAMPLE 8 Verification of an Implicit Solution

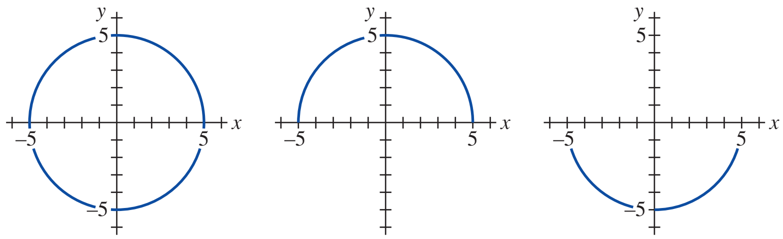
The relation $x^2 + y^2 = 25$ is an implicit solution of the nonlinear differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{8}$$

on the interval defined by $-5 < x < 5$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0. \tag{9}$$

Solving the last equation in (9) for the symbol $dy/dx$ gives (8). Moreover, solving $x^2 + y^2 = 25$ for $y$ in terms of $x$ yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $(-5, 5)$. The solution curves given in **FIGURE 1.1.2(b)** and 1.1.2(c) are segments of the graph of the implicit solution in Figure 1.1.2(a).




(a) Implicit solution $x^2 + y^2 = 25$

(b) Explicit solution $y_1 = \sqrt{25 - x^2}, -5 < x < 5$

(c) Explicit solution $y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

**FIGURE 1.1.2** An implicit solution and two explicit solutions in Example 8

Any relation of the form $x^2 + y^2 - c = 0$ *formally* satisfies (8) for any constant $c$. However, it is understood that the relation should always make sense in the real number system; thus, for example, we cannot say that $x^2 + y^2 + 25 = 0$ is an implicit solution of the equation. Why not?

Because the distinction between an explicit solution and an implicit solution should be intuitively clear, we will not belabor the issue by always saying, "Here is an explicit (implicit) solution."

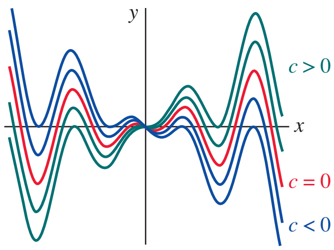



**FIGURE 1.1.3** Some solutions of $xy' - y = x^2 \sin x$

## Families of Solutions

The study of differential equations is similar to that of integral calculus. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant $c$ of integration. Analogously, when solving a first-order differential equation $F(x, y, y') = 0$, we *usually* obtain a solution containing a single arbitrary constant or parameter $c$. A solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ of solutions called a **one-parameter family of solutions**. When solving an $n$th-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions corresponding to the unlimited number of choices for the parameter(s). A solution of a differential equation that is free of arbitrary parameters is called a **particular solution**. For example, the one-parameter family $y = cx - x\cos x$ is an explicit solution of the linear first-order equation $xy' - y = x^2 \sin x$ on the interval $(-\infty, \infty)$ (verify). **FIGURE 1.1.3**, obtained using graphing software, shows the graphs of some of the solutions in this family. The solution $y = -x\cos x$, the red curve in the figure, is a particular solution corresponding to $c = 0$. Similarly, on the interval $(-\infty, \infty)$, $y = c_1e^x + c_2xe^x$ is a two-parameter family of solutions (verify) of the linear second-order equation $y'' - 2y' + y = 0$ in part (b) of Example 6. Some particular solutions of the equation are the trivial solution $y = 0$ $(c_1 = c_2 = 0)$, $y = xe^x$ $(c_1 = 0, c_2 = 1)$, $y = 5e^x - 2xe^x$ $(c_1 = 5, c_2 = -2)$, and so on.

In all the preceding examples, we have used $x$ and $y$ to denote the independent and dependent variables, respectively. But you should become accustomed to seeing and working with other symbols to denote these variables. For example, we could denote the independent variable by $t$ and the dependent variable by $x$.

### EXAMPLE 9 Using Different Symbols

The functions $x = c_1\cos 4t$ and $x = c_2\sin 4t$, where $c_1$ and $c_2$ are arbitrary constants or parameters, are both solutions of the linear differential equation

$$x'' + 16x = 0.$$